Briefing Facts on Digital Signal Processing

Hisakazu Kikuchi
Niigata University

revised on April, 2010

How to Use

This leaflet is a collection of short sentences that describe many facts on digital signal processing. They are a summary of issues taught in my course instruction. They are presented to help your fast and deep understanding of the theory of digital signal processing. You can refer them as a summary of DSP. You can learn many good textbooks on DSP in detail, and you can check by yourself whether you have gained a sufficient level of understanding on individual topics by making a reference to the briefing facts in this leaflet.

1 A Digital Signal Is Obtained by Sampling and Quantization

Sampling of a continuous-time signal produces a discrete-time signal. Quantization of the sampled values produces digital values. Sampled values are real-valued, whereas digital values are expressed by integers. A continuous quantity is referred to as analog, while a discrete quantity is referred to as digital.

The production process of a digital signal from a given analog signal consists of a pair of processing: sampling and quantization. Sampling is a discretization of the time (or space). Quantization is a discretization of the amplitude.

When a continuous-valued signal $x(n)$ is qunatized into a discrete-valued signal $y(n)$, a quantization error $e(n)$ is introduced. The process is represented by a linear stochastic model:

$$y(n) = x(n) + e(n).$$

Consider a uniform quantization in which the quantization step is Δ. Assume that $e(n)$ is a stochastic signal having a uniform distribution, and its *probability density function* (pdf) is hence written by $p(e) = \frac{1}{\Delta}$. The mean μ and the variance σ^2 of the quantization error are calculated as

$$\mu = \int_{-\infty}^{\infty} ep(e)de = \int_{-\Delta/2}^{\Delta/2} e\frac{1}{\Delta}de = \frac{1}{2\Delta}e^2\Big|_{-\Delta/2}^{\Delta/2} = 0,$$
$$\sigma^2 = \int_{-\infty}^{\infty} e^2p(e)de = \int_{-\Delta/2}^{\Delta/2} e^2\frac{1}{\Delta}de = \frac{1}{3\Delta}e^3\Big|_{-\Delta/2}^{\Delta/2} = \frac{\Delta^2}{12}.$$

In other words, the quantization error is expressed by a uniform distribution of a random signal of which mean and variance are 0 and $\dfrac{\Delta^2}{12}$, respectively.

If the quantization step is reduced to a half, the accuracy of $y(n)$ is improved by 1 bit. Then the level of the quantization error changes by

$$10\log\frac{\left(\frac{\Delta}{2}\right)^2/12}{\Delta^2/12} = 10\log\frac{1}{2^2} = 10\times(-2)\log 2 = -6\text{dB}.$$

If $y(n)$ is represented in a 20-bit accuracy, the dynamic range of $y(n)$ is $20\times 6 = 120$dB.

2 The Spectrum of a Discrete-Time Signal Is Periodic

2.1 Basic Facts

(1) Remember that **a function is a vector**. A signal is a vector.
However, a vector is not always a function.

(2) If the *inner product* between two vectors, a and b, is zero, they are orthogonal to each other.

$$\langle a, b\rangle = 0 \Leftrightarrow a \perp b$$

(3) The exponential function, $e^{j\omega t}$, forms an orthogonal system.

$$\Leftrightarrow \left\langle e^{j\omega_1 t}, e^{j\omega_2 t}\right\rangle = 2\pi\delta(\omega_1 - \omega_2), \quad \text{where } \delta(x-y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise} \end{cases}$$

(4) $e^{j\omega t}$ is periodic with the period of $T = \dfrac{2\pi}{\omega}$.

(5) A product of the period and the frequency of a periodic function is 1.

(6) Differentiation is a linear operator: $\dfrac{d}{dt}e^{j\omega t} = j\omega e^{j\omega t}$.
$e^{j\omega t}$ is an *eigen vector* of differentiation. $j\omega$ is the *eigen value* of differentiation.

(7) A *projection* P is defined by an operator such that $P^2 x = Px$, where x is a vector.

(8) The *Fourier transform* of a function is defined by the projection of the function onto the exponential function (which is an orthogonal basis vector). The projection is computed by an inner product.

$$X(\omega) = \left\langle e^{j\omega t}, x(t)\right\rangle$$

(9) *Parseval's Theorem:* Consider a square-integrable function space where $\{\beta_n(t)\}$ denotes a basis system. For any functions of $f(t)$ and $g(t)$,

$$\big\langle f(t),\, g(t)\big\rangle = \sum_n \big\langle \beta_n(t),\, f(t)\big\rangle^* \big\langle \beta_n(t),\, g(t)\big\rangle,$$

where * denotes complex conjugate.

- If $\beta_n(t) = e^{j\omega t}$ and if $f(t) = g(t)$, it reduces as follows.
$\displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$, where $F(\omega)$ is the Fourier transform of $f(t)$.

(10) In the case of continuous-time signals, *the delta function*[1] forms an orthogonal system.

$$\big\langle \delta(t_1),\, \delta(t_2)\big\rangle = \begin{cases} 2\pi, & \text{if } t_1 = t_2 \\ 0, & \text{otherwise} \end{cases}$$

(11) The Fourier transform of the delta function is 1.

$$\varDelta(\omega) = \left\langle e^{j\omega t}, \delta(t)\right\rangle = 1$$

(12) In the case of discrete-time signals, *the unit impulse* forms an orthogonal system.

$$\langle \delta(m),\, \delta(n)\rangle = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise} \end{cases}$$

(13) A discrete-time signal is expanded by a sequence of the unit impulses.

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

(14) A signal given in the time domain can be represented in the frequency domain. It is referred to as a *frequency spectrum.*

[1] Mathematically, the delta function is a *distribution* rather than an ordinary function. It is defined by the following equation. $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$, for any function $f(t)$ and any constant, a.

(15) Since a periodic function is expanded in a *Fourier series*, its spectrum is discrete.

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j\frac{2\pi}{T}kt}$$

$$\text{where } X_k = \frac{1}{T}\left\langle e^{j\frac{2\pi}{T}kt},\, x(t)\right\rangle,$$

and T is the period. Note that t denotes real numbers and k denotes integers. Identify what is continuous or discrete. Identify what is periodic or non-periodic.

(16) Two worlds of time and frequency are *dual.* A pair of variables of t and ω are exchangeable in $e^{j\omega t}$ and are mathematically in equal parts.

(17) As a consequence, we obtain the following facts.

- If the waveform of a signal is periodic in the time domain, the spectrum is discrete in the frequency domain.
  - It is understood from the Fourier series.
  - If the period is T, the spectral spacing between two adjacent frequency components is $\frac{1}{T}$ Hz.
- On the contrary, if a spectrum is discrete, the waveform of the signal is periodic.
- If the waveform of a signal is discrete in the time domain, the spectrum is periodic in the frequency domain.
  - It is understood from the duality of time and frequency.
  - If the sampling period[2] is T_s, the spectral period is $\frac{1}{T_s}$ Hz.
- On the contrary, if a spectrum is periodic, the waveform of the signal is discrete.

2.2 Sampling Theorem

Assume that a continuous-time signal is band-limited within an interval of $[-W, W]$ Hz in the frequency domain. The original signal is perfectly recovered from a sequence of sampled values, if the sampling frequency is $2W$ Hz or higher. [3]

Remember that a given continuous-time signal is not always band-limited. To avoid unwanted *aliasing*, an anti-aliasing lowpass filter must be applied to the continuous-time signal before sampling.

2.3 Discrete Fourier Transforms

A review on $e^{j\theta}$: It is a complex number on *the unit circle* in the complex plane. θ represents the angle from the real axis in the anti-clockwise direction. The magnitude of $e^{j\theta}$ is one, since it locates on the unit circle. $e^{j\theta}$ travels on the unit circle and comes back to 1, when θ increases from 0 through 2π [radian]. Hence it is a periodic function with period of 2π. Note that 2π [radian] is equivalent to 360 degrees. The projection of $e^{j\theta}$ onto the real axis is expressed by $\cos\theta$, and the projection onto the imaginary axis is expressed by $j\sin\theta$. Hence $e^{j\theta} = \cos\theta + j\sin\theta$.

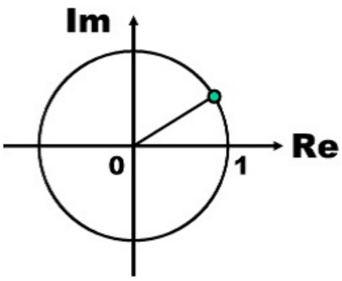

Fig. 1: A complex number on the unit circle.

[2] the spacing between two consecutive samples

[3] It was independently proved by Shannon and Someya in 1949. C.E. Shannon, "Communications in the presence of noise," *Proc. IRE,* vol. 37, no.1, pp. 10-21, Jan. 1949. I. Someya, *Waveform Transmission,* Shukyo-sha, Tokyo, 1949.

A pair of definitions of 4-point DFT and IDFT are given as follows.

$$\begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix} = \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^{-1} & W^{-2} & W^{-3} \\ W^0 & W^{-2} & W^{-4} & W^{-6} \\ W^0 & W^{-3} & W^{-6} & W^{-9} \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix}, \tag{1}$$

$$\begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{pmatrix} = \frac{1}{4} \begin{pmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & W^{1} & W^{2} & W^{3} \\ W^0 & W^{2} & W^{4} & W^{6} \\ W^0 & W^{3} & W^{6} & W^{9} \end{pmatrix} \begin{pmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{pmatrix}, \tag{2}$$

where $W = e^{j\frac{2\pi}{4}} = j$ and is referred to as the twiddle factor[4] of 4-point DFT. You will see the regular structure involved with DFT and IDFT in the matrix forms. Furthermore, if Eq. (1) is substituted into Eq. (2), the perfect reconstruction will be proved.

It is straitforward to write a general N-point DFT and IDFT, as follows.

$$\begin{pmatrix} \vdots \\ X(k) \\ \vdots \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & W^{-kn} \cdots & \\ & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ x(n) \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ x(n) \\ \vdots \end{pmatrix} = \frac{1}{N} \begin{pmatrix} & \vdots & \\ \cdots & W^{nk} \cdots & \\ & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ X(k) \\ \vdots \end{pmatrix}$$

A typical frequency spectrum of an 8-point DFT is shown in Fig. 2. It is important to understand the periodicity and the alignment of the spectrum. The frequency index, $k = [0,\ 7]$, is a one-to-one mapping of each point on the unit circle, which is written by $W^k = e^{j\frac{2\pi}{8}k}$. Since the spectrum is periodic with period of 8, $X(k - 8m) = X(k)$ for any integer, m. Also, note that the lowest frequency corresponds to the point of $e^{j2\pi \cdot 0} = 1$ on the unit circle, and the highest frequency corresponds to the point of $e^{j\frac{2\pi}{2}} = -1$ on the unit circle, respectively, in ordinary digital signal processing.[5]

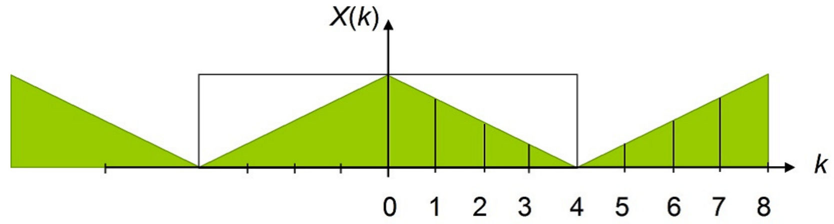

Fig. 2: Spectrum and the frequency index in the 8-point DFT.

DFT can be efficiently computed by an FFT (*fast Fourier transform*) algorithm. N^2 multiplications in N-point DFT are reduced to $N \log_2 N$ in FFT.[6] FFT is computed *in-place*: the original data sequence $x(n)$ is over-written by the transformed data, $X(k)$, and hence no additional memory is needed for storing $X(k)$.

2.4 Four Different Fourier Representations

Note that DFT is a representation for a *finite-length* discrete-time signal. On the other hand, there are 4 kinds of Fourier representations for *infinite-length* signals.

[4]Conventionally it is written by $W = e^{-j\frac{2\pi}{4}}$ in many textbooks. However the author prefers the above, because the conventional notation is opposite to that in the Fourier series. It does not matter which is used.

[5]In multi-rate signal processing, frequency bands can be extended or shrinked.

[6]If $N = 1024$, $N^2 \simeq 10^6$ and $N \log_2 N \simeq 10^4$. Imagine a 2-dimensional case such as in image processing.

(1) *Fourier series* is applied to a periodic continuous-time signal.

(2) *Fourier transform* is applied to a non-periodic continuous-time signal.

(3) *Discrete Fourier series* is applied to a periodic discrete-time signal.

(4) *Discrete-time Fourier transform* is applied to a non-periodic discrete-time signal.

2.5 The Wider a Window Opens, the More Details You Can See

A window function is a popular tool in spectral analysis of a digital signal. It is used to make a focus on significant frequency components of interest. Fine spectral resolution and large attenuation of undesirable frequency bands are demanded.

(1) A *window function* is a lowpass function.

(2) If the aperture time (time duration) of a window function is *long*, the bandwidth of the main lobe is narrow in the frequency domain and the *frequency resolution* is fine.

(3) If the waveform of the aperture of a window *decays smoothly*, the heights of the side lobes decrease in the frequency domain and the *spectral leakage* also decreases.

The main lobe in the frequency spectrum of a window function corresponds to the significant band for passing the wanted signal components. The side lobes correspond to insignificant frequency components to be filtered out. The ratio of the main lobe height to the height of the 2nd lobe (the highest side lobe) is desired as large as possible. The requirement on the shortness of a window width and the smoothness of the aperture and that on fine frequency resolution and less spectral leakage are subject to trade-off.

3 Linear Discrete-Time Systems

3.1 Description by Impulse Responses and Convolutions

We consider a digital signal processing system that is linear and translation-invariant.

(1) A system is said to be *linear*, if the superposition holds.

(2) A system is said to be *translation-invariant*, if the property of the system is unchanged when the system is translated.

- Assume that a property of a system is represented by an equation of $f(t) = 0$ in the time domain.
  - If it is *time-invariant*, then $f(t-a) = 0$ for any constant, a.
- Assume that a property of a system is represented by an equation of $g(x) = 0$ in the spatial domain.
  - If it is *space-invariant*, then $g(x-a) = 0$ for any constant, a.
- Assume that the input and output of a system are described by $y(t) = L\{x(t)\}$.
  - If the system L is *time-invariant*, then $y(t-a) = L\{x(t-a)\}$ for any constant, a.

(3) The output response of a system to the unit impulse is referred to as the *impulse response.*

$$h(n) = \sum_{k=-\infty}^{\infty} h(n-k)\delta(k)$$

(4) The output response $y(n)$ of a linear translation-invariant system is given by a convolution of the impulse response $h(n)$ and the input $x(n)$.

$$y(n) = h(n) * x(n) = \sum_k h(n-k)x(k)$$

(5) The z-transform of $x(n)$ is defined by $X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$.

(6) Applying the z-transform to both sides of the input-output equation, one obtains

$$Y(z) = H(z)X(z).$$

A convolution of $h(n)$ and $x(n)$ in the time domain is converted to a product of $H(z)$ and $X(z)$ in the z domain.

(7) Remark: The inverse DFT of a product of two DFTs, $H(k)$ and $X(k)$, is different from $y(n) = h(n) * x(n)$. That is,

$$\mathrm{IDFT}\{H(k)X(k)\} \neq h(n) * x(n).$$

The left-hand side is a cyclic convolution of $h(n)$ and $x(n)$, that is defined on periodic sequences. The right-hand side is a linear convolution of $h(n)$ and $x(n)$, that is defined on non-periodic infinite-length sequences.

(8) z is the discrete-time complex frequency.

(9) $H(z)$ is referred to as a transfer function.

(10) The frequency response is expressed by $H(\omega) = H(z)|_{z=e^{j\omega}}$. That is, the frequency response is the values of the transfer function on the unit circle.

(11) A point where $H(z) = 0$ is referred to as a zero.

(12) A point where $H(z) \to \infty$ is referred to as a pole.

(13) The necessary and sufficient condition for a system to be stable is that no pole exists on or outside the unit circle.

3.2 Analysis of Linear Translation-Invariant Systems

To compute the response of a system to a given input signal, the following sequence of procedures is applied.

(1) Define all output nodes of delay elements as state variables: label them as in $q(n)$, $q(n-1)$, $q(n-2)$, and so on.

(2) Write a set of state equations that are described by linear combinations of state variables as well as inputs and outputs.

(3) Apply the z-transform to every state equation.

(4) Solve a set of transformed equations to obtain the transfer function.

(5) Find the locations of poles, and check if the system is stable.

(6) Apply the inverse z-transform to $Y(z) = H(z)X(z)$ to obtain the output response $y(n)$.

- Make a partial fraction expansion of $H(z)X(z)$ to write it in a linear combination of partial fractions with respect to individual poles.
- Use the following inversion formula to obtain the inverse z-transform of individual partial fractions.
- a geometric series (exponential signal): p^n, for $n \geq 0 \longleftrightarrow \dfrac{1}{1 - pz^{-1}}$
- the unit impulse: $\delta(n) \longleftrightarrow \mathit{\Delta}(z) = 1$
- the unit step: $u(n) \longleftrightarrow U(z) = \dfrac{1}{1 - z^{-1}}$

3.3 Linear-Phase FIR Filters

Digital filters are categorized in two groups of finite impulse response (FIR) filters and infinite impulse response (IIR) filters.

(1) Linear phase implies that the phase of a system or a signal varies in a linear function of the frequency.

(2) The linear-phase property is realized by an FIR filter.

(3) The impulse response of an FIR filter is either symmetric or anti-symmetric.

(4) If a linear-phase FIR filter is of odd-length in time duration, the center of these symmetries locates on the center of the impulse response.

(5) If a linear-phase FIR filter is of even-length, the center of these symmetries is off the sample points and locates on the mid point of the impulse response.

An IIR filter has a feedback loop, and hence it is referred to as recursive structure. In contrast, if a system has no feedback loops, it is referred to as non-recursive structure and results in an FIR filter. However, a feedback structure does not always result in an IIR filter, because the denominator factors caused by the feedback loops may cancel out with the numerator factors in the transfer function. It is a kind of pole-zero cancellation.

4 Adaptive Filters

4.1 Define the error and minimize it

In adaptive filtering, a desired signal (or a reference signal) is given as shown in Fig. 3. The input signal is usually unknown. The response of the adaptive filter to the input is subtracted from the desired signal to generate the error signal. The adaptive filter is controlled by a particular algorithm so that the error may approach as small as possible.

When the update becomes to be negligibly small, the adaptive filter is said to have converged. Once the convergence is established, the response of the adaptive filter mimics the desired signal. In other words, if the desired signal is the impulse response of an unknown system, and if the input is specified by the unit impulse, the adaptive filter can be a good model to the unknown system.

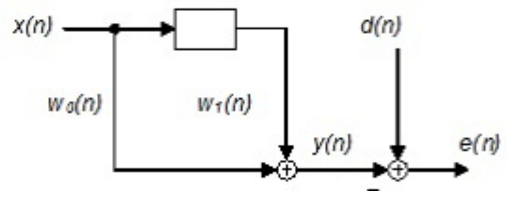

Fig. 3: 1st-order adaptive filter.

Let us consider a basic problem. The input signal $x(n)$ is assumed to be stationary. We want to minimize the MSE (mean square error)

$$\varepsilon = E[e^2(n)],$$

where $E[\cdot]$ denotes the expectation value and $e(n) = d(n) - y(n)$ is referred to as the error signal. The MSE is evaluated after a substitution of

$$y(n) = \sum_{k=0}^{M-1} w_k(n)x(n-k)$$

into $e(n)$. $w_k(n)$ is the kth coefficient at present. Since ε is a 2nd-order (parabolic) function of $w_k(n)$ and is convex downward, ε takes its minimum at a point where the gradient of ε is equal to 0. As a result, we obtain

$$\frac{\partial \varepsilon}{\partial w_k(n)} = -2E[e(n)x(n-k)] = 0.$$

This is a linear equation system where M unknowns are $w_k(n)$. The solution will be optimum in the sense of minimum MSE. Practically it is difficult to compute the expectation value, because $x(n)$ is an infinitely-long random signal. Even if $x(n)$ is stationary, the ensemble average instead of the expectation value requires the entire set of sample values over $x(n)$.

A feasible way to the optimal solution is a step-by-step approach. Note that the surface of ε is an M-dimensional parabola being convex downward. Hence, if an instantaneous estimate of the gradient is obtained, ε can travel along the direction of the gradient estimate to reach the minimum point. The filter coefficients of an adaptive filter are thus updated sample by sample.

$$w_k(n+1) = w_k(n) + \Delta w_k(n),$$

where $\Delta w_k(n)$ is the update for setting the coefficient value at the next time instance.

4.2 The update is given by a product of the error and the input

Steepest Descent: A straightforward algorithm for setting the instantaneous update is as follows.

$$\Delta w_k(n) = \mu E[e(n)x(n-k)],$$

where μ is a small constant to ensure the convergence.[7] This is referred to as the steepest descent method, but it suffers from a huge number of calculations for the ensemble average.

[7] A typical value of μ is 10^{-6}. It depends on actual problems.

LMS: The simplest approximation of the expectation value can be the present value of $e(n)x(n-k)$. That is, the coefficient update is set in proportion to a product of the error signal and the input signal which is available at the kth delay element. In LMS (least mean square) algorithm which is most popular, the update equation is given by

$$\Delta w_k(n) = \mu e(n)x(n-k).$$

Its disadvantage is that the update depends on the input. When the input is small in amplitude, the update can be too small, even if the magnitude of the error is still large. On the contrary, when the input is too large, the error can be over-estimated.

Normalized LMS: In this algorithm, the update in LMS is normalized by the mean square of the input signal.

$$\Delta w_k(n) = \mu \frac{e(n)x(n-k)}{E[x^2(n-k)]}$$

Clipped LMS: A further simplification can be developed. Instead of $x(n-k)$, its polarity between positive and negative is only used so that the algorithm is independent of the magnitude of the input.

$$\Delta w_k(n) = \mu e(n)\,\mathrm{sgn}[x(n-k)],$$

where $\mathrm{sgn}[\cdot]$ implies the sign of the argument.

RLS: When the coefficient values are updated, a posteriori error, $e(n) = d(n) - \langle \vec{w}(n), \vec{x}(n) \rangle$, is used in the previous algorithms. In contrast, a priori error, $e(n) = d(n) - \langle \vec{w}(n-1), \vec{x}(n) \rangle$, is used in RLS (recursive least square) algorithm. RLS is fast at the expense of increased complexity.

- Initialize:
  - Set λ such that $0 \ll \lambda < 1$. For example, $\lambda = 0.9$.
  - Set $P(0) = \frac{1}{\delta} I$, where I is the $M \times M$ identity matrix and $0 < \delta \ll 1$, such as $\delta = 0.1$.
  - Set $\vec{w}(n) = 0$.
- Repeat {

  $\vec{r}(n) = P(n-1)\vec{x}(n)$

  $c(n) = \langle \vec{r}(n), \vec{x}(n) \rangle + \lambda$

  $\vec{a}(n) = P(n-1)\vec{x}(n)/c(n)$

  $e(n) = d(n) - \langle \vec{w}(n-1), \vec{x}(n) \rangle$

  $\vec{w}(n) = \vec{w}(n-1) + e(n)\vec{a}(n)$

  $Q = \vec{a}(n)\ \vec{r}^{\dagger}(n)$, where † denotes transposition and Q is an $M \times M$ matrix.

  $P(n) = \{P(n-1) - Q\}/\lambda$

  }

Typical applications of adaptive filtering are illustrated in Fig. 4. They are interference cancellation, prediction, identification, and inverse modeling.

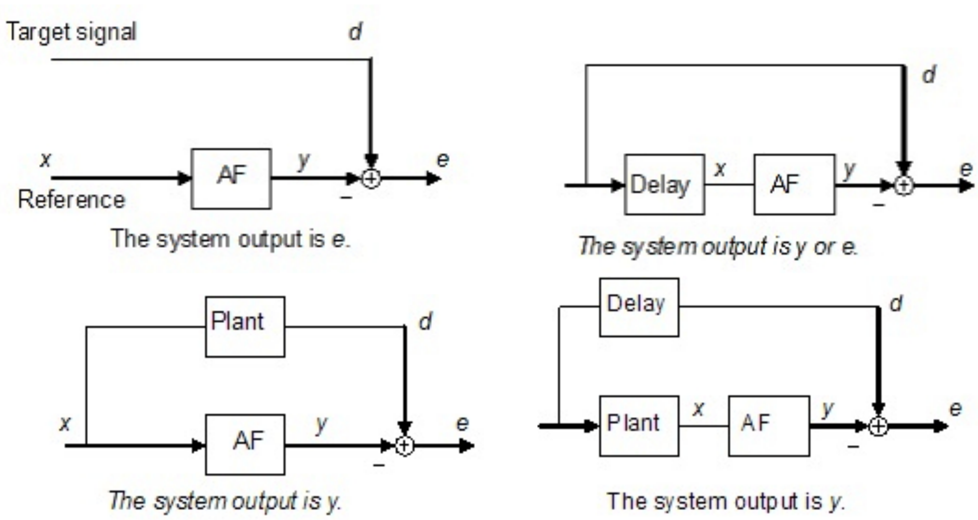

Fig. 4: Typical applications of adaptive filters. Top-left: interference cancellation, top-right: prediction, bottom-left: identification, and bottom-right: inverse modeling.

Appendix

A Time and frequency in a broad sense of the terms

In the physical worlds, there are signals that are described by functions of time, particularly of the continuous time. Speech signals and acoustic signals are popular examples. In addition, there are different signals described in the spatial domain. For example, a photograph is represented by the intensity distribution over a 2-dimensional space. A video signal is expressed by a function of space and time.

In digital signal processing, a signal is sampled and is represented by a set of discrete data. According to its mathematical nature, digital signals generally lose their original physical characterizations. A digital signal is just a sequence of numerical data. In this way, a speech signal is represented by a sequence of one-dimensional data, and an image is represented by a sequence of two-dimensional data.

Hence the term of *time* can be used to describe a sequence of data. It means the actual time index for time-domain signals. On the other hand, it means a two-dimensional space for images in a broad sense.

B Time and frequency are a pair of cyclic coordinates

The following topics are loosely related to signal processing. However, the author believes that they may be some triggers to understand the real physical worlds even if many phenomena are treated by digital signal processing.

- A conservation law holds in an arbitrary linear and translation-invariant dynamical system.
  - If a dynamical system is linear and time-invariant, the *energy conservation law* holds.
  - If a dynamical system is linear and space-invariant, the *momentum conservation law* holds.
- Such a dynamical system obeys to the canonical equations[8] in the classical analytic dynamics. The canonical equations are written by a pair of *canonical coordinates.*

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

  q_i and p_i denote the position and the momentum of a particle in a 3-dimensional space, and they are referred to as canonical coordinates. $\dot{}$ stands for the derivative with respect to time. H is the Hamiltonian of the dynamical system.
- Time and energy are a pair of *cyclic coordinates.* Position and momentum are a pair of cyclic coordinates. An *uncertainty relationship* holds between a pair of cyclic coordinates in quantum mechanics.

$$\Delta E \cdot \Delta t \geq \frac{1}{2}\hbar, \quad \Delta p_i \cdot \Delta q_i \geq \frac{1}{2}\hbar$$

  On a quantum particle such as electron, photon, and phonon, $E = \hbar\omega$ and $p_i = \hbar k_i$, where $\hbar$ is the Planck's constant divided by 2π and k_i is a *wave number.* That is, the frequency is equivalent to the energy and the wave number is equivalent to the momentum. The uncertainty relationships are rewritten as

$$\Delta\omega \cdot \Delta t \geq \frac{1}{2}, \quad \Delta k_i \cdot \Delta q_i \geq \frac{1}{2}.$$

- Similar quantities appear in a plane wave, $e^{j(\omega t - kx)}$. The wave travels along the x direction with time in the propagation of electro magnetic waves and acoustic waves. A sinusoidal wave involves the products of the cyclic coordinate pairs on its shoulder.

Power is defined by the instantaneous incremental energy with respect to time.

[8] A pair of them is the most beautiful equations in physics.

C Cutset Scaling of a Signal Flow Graph

Given a *signal flow graph* that represents a linear translation-invariant system by using nodes and arrows, a cutset is defined by a simple closed curve that encircles at least one node. (See Fig. 5, where the cutset is drawn in a dashed circle.) A cutset separates a flow graph into two graphs of inside and outside. Hence there exists an equation of flows adding up to zero across the boundary.

Any branch weight, say $\alpha \neq 0$, among constant-multiplication, delay, and transfer function along an outgoing branch can be changed into unity in such a way that it is shifted from the incoming branches by multiplying by α^{-1} and onto outgoing branches by multiplying by α, respectively. In this way, an equivalent representation of a signal flow graph is obtained.

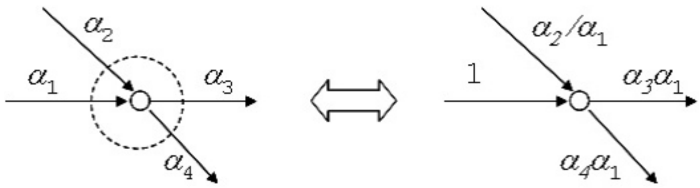

Fig. 5: Cutset scaling by α_1.

The concept of a cutset is an extension of a node in electric circuit theory. The cutset scaling is equivalent to the Kirchhoff's current law such that all incoming currents to a node add up to zero.